Int. J. Solids Structures, 1972, Vol. 8, pp. 913 to 921. Pergamon Press. Printed in Great Britain

AN ELASTIC–VISCOPLASTIC SOLUTION FOR IMPULSIVELY LOADED RINGS†

THOMAS DUFFEY

Sandia Laboratories,
Albuquerque, New Mexico

Abstract—Solutions are developed in terms of elementary functions for the transient and final deformation of an impulsively loaded, uniformly expanding ring. The ring is assumed to be fabricated of an elastic–perfectly plastic material that exhibits realistic nonlinear strain rate sensitivity. A numerical example of a strain rate sensitive mild steel ring is presented and compared with the corresponding solution in which strain rate sensitivity is ignored. The differences emphasize that the effects of strain rate sensitivity can be quite significant. Conclusions of this study lend support to an approximate technique suggested by Perrone.

INTRODUCTION

THE high energy forming of thin-walled circular rings has received considerable attention over the past few years for several different purposes. A decade ago, Johnson *et al.* [1] developed a method of imparting a uniform internal impulse to a circular ring. By observing the axisymmetric, transient expansion of the ring, dynamic stress–strain properties could be deduced. More recently, Hoggatt and Recht [2] improved upon the technique and simplified the data reduction necessary by making an experimental observation on the deceleration of the ring following the initial impulse. Ching and Weese [3] investigated a number of techniques for explosive forming of thin rings. They developed detailed analyses to account for the transient response of a pressure-pulse loaded ring and successfully correlated the analysis with experiments using a ring constructed of a strain rate insensitive material. However, many materials, such as mild steel, are known to exhibit significant strain rate sensitivity [4, 5], and it is unclear what influence this will have on the ring response, since the analysis of Ching and Weese [3] does not take this effect into account.

Perrone [6, 7] obtained a solution for nonlinearly strain rate sensitive ring response to impulsive loading. However, the analysis is based upon a rigid–plastic material assumption. Further, only the final response of the ring is predicted. Transient and final solutions for the elastic–plastic response of rings to impulsive loading are presented in [8], but a somewhat unrealistic linear stress–strain rate relation was utilized to obtain solutions.

It is the purpose of this paper, then, to present solutions for the transient and final deflection of a uniformly expanding, impulsively loaded ring that are not restricted by the simplifying material assumptions in Refs. [6–8], and yet are not too complex to be useful for design purposes.

† This work was supported by the United States Atomic Energy Commission, and was presented at the Third International Conference of the Center for High Energy Forming, Vail, Colorado, July 12–16, 1971.

In this paper, the dynamic yield stress is assumed to vary with rate of straining in the following well-known manner,

$$\frac{\sigma_y^D}{\sigma_y} = 1+\left(\frac{\mathrm{d}\varepsilon/\mathrm{d}t}{D}\right)^{1/p} \tag{1}$$

where D and p are material constants, $\mathrm{d}\varepsilon/\mathrm{d}t$ denotes strain rate, σ_y^D is dynamic yield stress and σ_y represents static yield stress. This nonlinear strain rate sensitivity behavior has been shown to approximately fit experimental strain rate sensitivity data for common materials [9], and is the same law as utilized by Perrone [6, 7].

In this paper, solutions for transient ring response are obtained by partitioning the solution into three phases, (1) an initial elastic phase before the circumferential stress in the ring has reached the strain rate sensitive yield stress, (2) a fully plastic phase and (3) an elastic unloading phase. In the second phase, the equation of radial motion is nonlinear and cannot be solved directly. The solution is obtained by appropriate changes of variables and rewriting the original second order equation as two first order equations, the first representing the variation of strain rate with strain and the second representing the variation of strain rate with time. Both of these equations are readily solved by integrating over the range in strain rates occurring. Explicit solutions are then written for displacement of the ring as a function of strain rate and for time as a function of strain rate, from which displacement as a function of time as well as peak displacement can be deduced. The solutions appear in the form of a finite series and the number of terms appearing is a function of the material constant p described earlier. The solutions are developed in terms of elementary functions and are useful for design purposes.

A numerical example of an impulsively loaded, strain rate sensitive mild steel ring is presented and compared to the corresponding solution based upon strain rate insensitivity (static yield stress). Differences in peak deflections between the two cases are shown to be significant in the range of initial strain rates examined of 1000–4000 sec^{-1}.

ANALYSIS OF THE PROBLEM

The radial equation of motion for a thin, impulsively loaded ring, Fig. 1, may be written as

$$\frac{\mathrm{d}^2 w}{\mathrm{d}t^2}+\frac{\sigma}{\rho a} = 0 \tag{2}$$

if only uniform radial expansion and small deflections are considered. In equation (2), w denotes outward radial displacement, ρ is mass density, a is radius and σ is circumferential stress in the ring. For uniform radial motion, the circumferential strain in the thin ring is related to radial displacement by

$$\varepsilon = w/a. \tag{3}$$

Material behavior is taken as elastic–perfectly plastic with elastic unloading. As shown in Fig. 2, the dynamic yield stress, σ_y^D, is assumed to be nonlinearly strain rate sensitive. The nonlinear power law relation between the dynamic yield stress and the strain rate is more fully discussed in the Appendix.

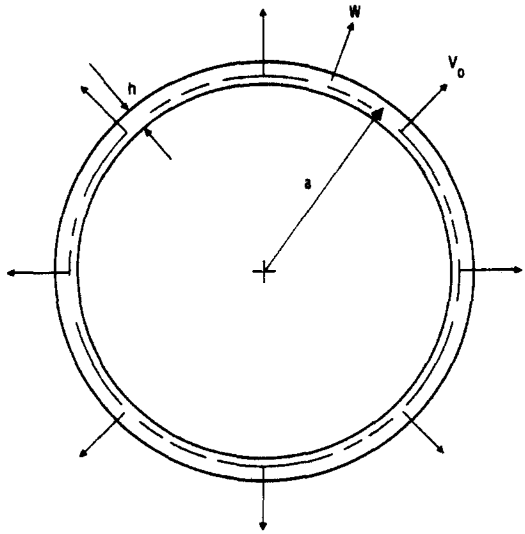

FIG. 1. Impulsively loaded ring.

Initial elastic phase

It is convenient to introduce the following nondimensional parameters,

$\tau = ct/a$ and $u = w/a$, where

$c = (E/\rho)^{1/2}$, the bar velocity and E is the elastic modulus.

Using Hooke's law, and equations (2) and (3), the following equation of motion may be developed:

$$\frac{d^2u}{d\tau^2}+u = 0. \tag{4}$$

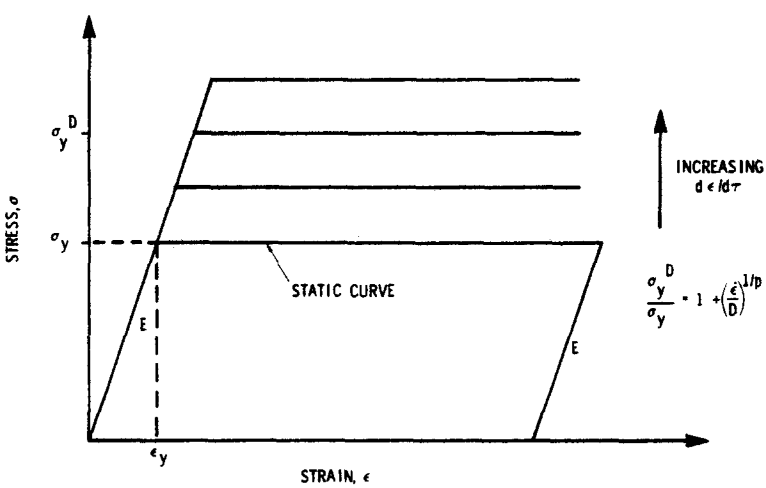

FIG. 2. Nonlinearly strain rate sensitive material behavior.

This equation is subjected to the initial conditions $u(0) = 0$ and $du/d\tau(0) = V_0/c$, where V_0 is the initial outward radial velocity. The solution is

$$u = \eta \sin \tau \tag{5}$$

where $\eta = V_0/c$.

This solution is applicable until the ring cross section reaches the dynamic yield stress. The dynamic yield stress is assumed to vary with rate of straining in the following power law manner, as mentioned earlier.

$$\frac{\sigma_y^D}{\sigma_y} = 1 + \left(\frac{d\varepsilon/dt}{D}\right)^{1/p} \tag{6}$$

where D and p are material constants, σ_y^D denotes dynamic yield stress and σ_y represents static yield stress. The time at which yielding first occurs may be determined by equating the stress calculated assuming an elastic path to the dynamic yield stress of equation (6),

$$\eta E \sin \tau_1 = \sigma_y[1 + \mu(\eta \cos \tau_1)^{1/p}] \tag{7}$$

where $\mu = (c/aD)^{1/p}$ and where τ_1 is the time at which yielding is initiated. An explicit solution for τ_1 does not appear feasible for most values of p.†

Plastic phase

The initial conditions on the motion in this phase are

$$\begin{aligned} u(\tau = \tau_1) &= u_1 = \eta \sin \tau_1 \\ \frac{du}{d\tau}(\tau = \tau_1) &= \frac{du_1}{d\tau} = \eta \cos \tau_1 . \end{aligned} \tag{8}$$

Omitting strain hardening, the following equation of motion may be developed for the rate-sensitive plastic response.

$$\frac{d^2u}{d\tau^2} + \mu\varepsilon_y\left(\frac{du}{d\tau}\right)^{1/p} + \varepsilon_y = 0. \tag{9}$$

Unfortunately, equation (9) is nonlinear and it does not appear possible to obtain a closed form solution for $u(\tau)$ by two direct integrations. However, an alternate procedure is available. With the substitution of $q = du/d\tau$, equation (9) can be written either as a first order differential equation that represents the variation in strain rate with strain or as a differential equation that represents the variation in strain rate with time. Since both these equations can be readily integrated, displacement as a function of time as well as peak displacement can be deduced.

Considering first the variation of strain rate with strain (nondimensional displacement), the plastic displacement can be determined by integrating over all strain rates occurring up to the displacement of interest,

$$u = \frac{-1}{\varepsilon_y}\int_{(d\varepsilon/d\tau)_i}^{(d\varepsilon/d\tau)} \frac{q\, dq}{(\mu q^{1/p} + 1)} + \eta \sin \tau_1 \tag{10}$$

† A solution for $p = 1$ (linear strain rate sensitivity) is presented in [8].

where $(d\varepsilon/d\tau)_i$ denotes the initial plastic strain rate, $\eta \cos \tau_1$, i.e. the strain rate at yielding.

The integrand of equation (10) may be transformed into rational form by a substitution $q = \chi^p$, and the result may be integrated [10] as

$$u = \frac{p}{\varepsilon_y}\left[\sum_{v=0}^{2p-2} \frac{(-1)^v}{(2p-1-v)\mu^{(v+1)}}\left\{(\eta \cos \tau_1)^{(2p-1-v)/p} - (d\varepsilon/d\tau)^{(2p-1-v)/p}\right\} + \frac{(-1)^{2p-1}}{\mu^{2p}} \ln\left\{\frac{\mu(\eta\cos\tau_1)^{1/p}+1}{\mu(d\varepsilon/d\tau)^{1/p}+1}\right\}\right] + \eta \sin \tau_1 \tag{11}$$

$$p = 1, \tfrac{3}{2}, 2, \tfrac{5}{2}, 3, \ldots,$$

$$\eta \sin \tau_1 \leq u \leq u_2$$

where u_2 represents the peak deflection, readily obtained from equation (11) for the condition $d\varepsilon/d\tau = 0$.

Using a substitution similar to that used above, equation (9) may be written as an explicit integral for time,

$$\tau = \tau_1 - \frac{1}{\varepsilon_y}\int_{(d\varepsilon/d\tau)_i}^{(d\varepsilon/d\tau)} \frac{dq}{\mu q^{1/p}+1}. \tag{12}$$

Equation (12) can be written in rational form and integrated as

$$\tau = \tau_1 + \frac{p}{\varepsilon_y}\left[\sum_{v=0}^{p-2} \frac{(-1)^v}{(p-1-v)\mu^{(v+1)}}\left\{(\eta \cos \tau_1)^{(p-1-v)/p} - (d\varepsilon/d\tau)^{(p-1-v)/p}\right\} + \frac{(-1)^{p-1}}{\mu^{p}} \ln\left\{\frac{\mu(\eta\cos\tau_1)^{1/p}+1}{\mu(d\varepsilon/d\tau)^{1/p}+1}\right\}\right] \tag{13}$$

$$p = 2, 3, 4, \ldots,$$

$$\tau_1 \leq \tau \leq \tau_2$$

where τ_2 is the time at which peak deflection is reached. τ_2 can be determined from equation (13) from the condition $d\varepsilon/d\tau = 0$.

Solutions in the plastic range have been presented in equations (11) and (13), respectively for displacement as a function of strain rate, $u = f(d\varepsilon/d\tau)$ and time as a function of strain rate, $\tau = g(d\varepsilon/d\tau)$. Unfortunately, it does not appear feasible to write $u = f(g^{-1}(\tau))$ explicitly. Displacement–time coordinate pairs can still be readily deduced, however, since strain rate decreases monotonically in the plastic range. There exists, then, a unique time, τ, and displacement, u, associated with each strain rate in the range

$$\eta \cos \tau_1 \leq d\varepsilon/d\tau \leq 0 \tag{14}$$

so that for various values of strain rate over this range, displacement–time pairs may be calculated from equations (11) and (13) without resorting to numerical methods of solution.

Elastic unloading phase

As shown in Fig. 2, the ring unloads elastically.

Equations (2) and (3) may be combined with Hooke's law to obtain the following equation of motion.

$$\frac{d^2u}{d\tau^2}+u+(\varepsilon_y-u_2) = 0. \tag{15}$$

With the initial conditions $u(\tau_2) = u_2$ and $du/d\tau\ (\tau_2) = 0$, the solution is

$$u = u_2+\varepsilon_y[\cos(\tau-\tau_2)-1] \tag{16}$$

and the final deflection is simply

$$u_f = u_2-\varepsilon_y. \tag{17}$$

DISCUSSION

The influence of nonlinear strain rate sensitivity on peak deflections is indicated in Fig. 3 for a mild steel ring over a range in initial strain rates from 1000 in./in./sec to above 4000 in./in./sec. The steel material chosen is one that exhibits extreme strain rate sensitivity (D = 40·4, p = 5·0). Five cases are considered in Fig. 3. Curve 1 is rate insensitive, with stress–strain relation based upon static data. Curve 2 represents nonlinear rate-sensitivity, in which dynamic stress in the plastic range depends upon the current strain rate according to equation (6). Comparison of these two curves suggests that strain rate sensitivity can significantly alter ring response. Curve 3 is a rate-insensitive case, in which the dynamic stress in the plastic range is assumed to be equal to the initial dynamic yield stress corresponding to the strain rate when yielding commences. It is seen that case 3 is a reasonable approximation to case 2, lending further validity to an earlier observation by Perrone [6, 7] for uniformly expanding rings. Deflections predicted by the rigid-plastic theory of Perrone [6] are plotted for comparison in Fig. 3 as well (curve 4).

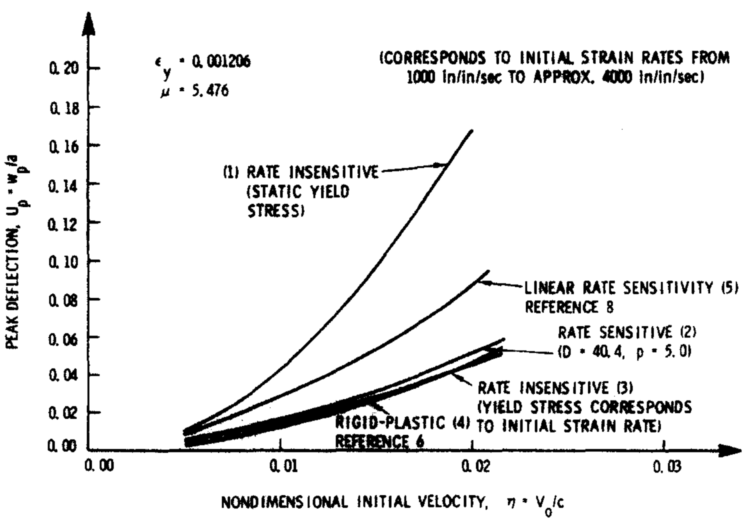

FIG. 3. Influence of nonlinear strain rate sensitivity on a mild steel ring.

Peak ring deflections, assuming linear strain rate sensitivity are shown as curve 5 in Fig. 3. It is seen that, at least for the highly nonlinear mild steel material considered in this example, a linear strain rate sensitivity approximation tends to underestimate the influence of rate sensitivity on deflections.

An example of the transient, elastic–plastic response of an impulsively loaded ring is shown in Fig. 4 for the parameters indicated on that figure. The three phases of motion are indicated for both the strain-rate sensitive and insensitive examples considered. The displacement–time coordinate pairs are determined utilizing equation (5) in the elastic range, equations (11) and (13) in the plastic range, and equation (16) in the elastic-unloading range.

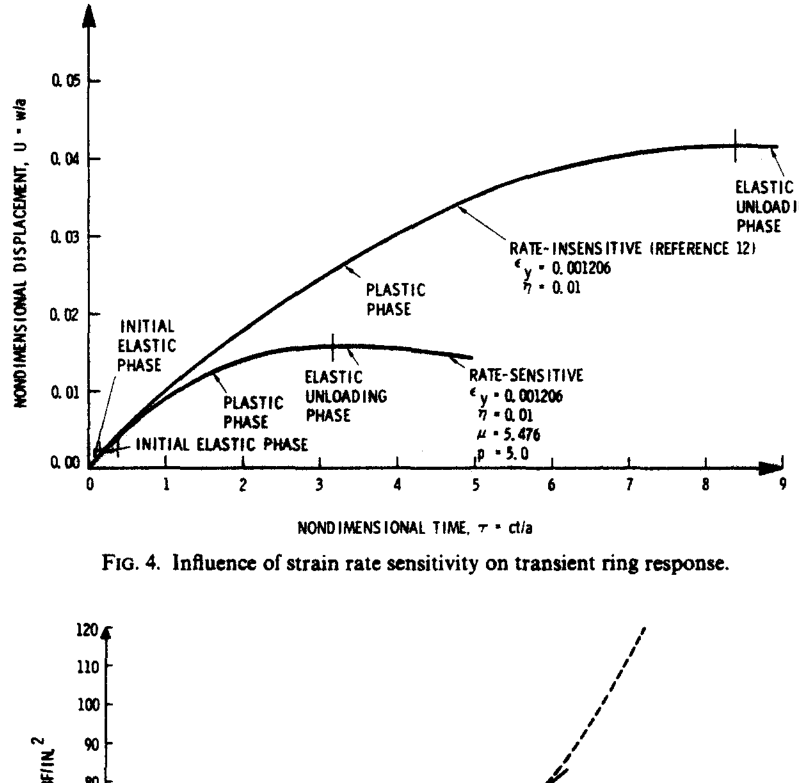

FIG. 4. Influence of strain rate sensitivity on transient ring response.

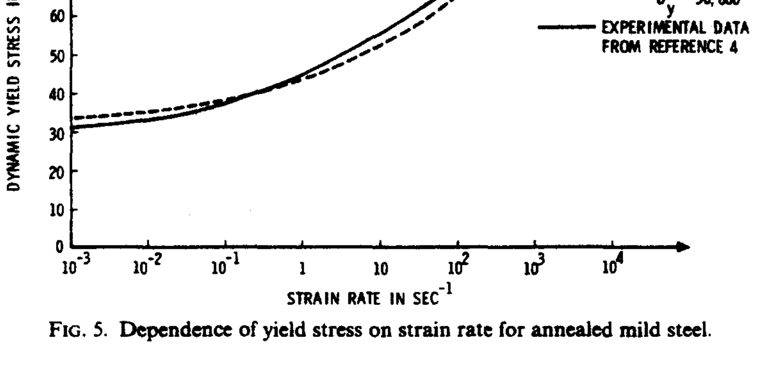

FIG. 5. Dependence of yield stress on strain rate for annealed mild steel.

RESULTS

Solutions for the transient and final responses of elastic–plastic rings subjected to a high intensity impulsive pressure have been developed. The solutions are presented in a form suitable for design calculations and only elementary functions appear.

A numerical example of a nonlinearly strain rate sensitive mild steel ring is presented and compared to the corresponding solution when strain rate sensitivity of the material is ignored. The results suggest that strain rate sensitivity should be included when analyzing the high energy forming of materials known to be strain rate sensitive.

REFERENCES

[1] P. C. JOHNSON, B. A. STEIN and R. S. DAVIS, Basic Parameters of Metal Behavior Under High Rate Forming, Arthur D. Little Inc. Report No. WAL TR 111.2/20-6 (1962).
[2] C. R. HOGGATT and R. F. RECHT, Stress–strain data obtained at high rates using an expanding ring. *Exp. Mech.* **9**, 441 (1969).
[3] L. K. W. CHING and J. A. WEESE, The Explosive Forming of Rings, *Proceedings, Second International Conference of the Center for High Energy Forming* (1969).
[4] M. J. MANJOINE, Influence of rate of strain and temperature on yield stresses of mild steel. *J. appl. Mech.* **11**, 211 (1944).
[5] C. J. MAIDEN and S. J. GREEN, Compressive strain rate tests on six selected materials at strain rates from 10^{-3} to 10^{4} in./in./sec. *J. appl. Mech.* **33**, 496 (1966).
[6] N. PERRONE, On a simplified method for solving impulsively loaded structures of rate-sensitive materials. *J. appl. Mech.* **32**, 489 (1965).
[7] N. PERRONE, Impulsively loaded strain hardened rate-sensitive rings and tubes. *Int. J. Solids Struct.* **6**, 1119 (1970).
[8] T. DUFFEY and R. KRIEG, The effects of strain-hardening and strain-rate sensitivity on the transient response of elastic–plastic rings and cylinders. *Int. J. Mech. Sci.* **11**, 845 (1969).
[9] S. R. BODNER and P. S. SYMONDS, Experimental and theoretical investigation of the plastic deformation of cantilever beams subjected to impulsive loading. *J. appl. Mech.* **29**, 719 (1962).
[10] W. GRÖBNER and N. HOFREITER, *Integraltafel*. Springer-Verlag (1965).
[11] N. JONES, Influence of strain hardening and strain rate sensitivity on the permanent deformation of impulsively loaded rigid-plastic beams. *Int. J. Mech. Sci.* **9**, 777 (1967).
[12] T. A. DUFFEY, Approximate solution of an impulsively loaded long cylinder governed by an elastic–plastic material law. *Acta Mech.* **11**, 45 (1971).

APPENDIX

The dynamic yield stress–strain rate relation, equation (6), is widely used [6, 7, 9, 11], and has been shown to approximately fit the observed behavior of some common engineering metals [9].

The well-known strain rate constants for steel, $D = 40{\cdot}4$ and $p = 5$, were originally determined in Ref. [9] based upon the earlier experimental work of Manjoine [4]. However, the correlation in Ref. [9] was only performed to a strain rate below $100\ \text{sec}^{-1}$.

Since higher strain rates than $100\ \text{sec}^{-1}$ often occur in uniformly expanding rings, the data of Manjoine is compared with the constants D and p indicated above to $1000\ \text{sec}^{-1}$. It is seen in Fig. 5 that at least to strain rates of $1000\ \text{sec}^{-1}$, $D = 40{\cdot}4$ and $p = 5{\cdot}0$ provide reasonable approximations to the actual material behavior.

(*Received* 12 *August* 1971; *revised* 27 *December* 1971)

Абстракт--Решается вопрос формулировки элементарной функции переходной и конечной деформации нагруженного импульсами, равномерно расширяюшегося кольца. Предполагается, что кольцо изготовлено из упругого совершенно пластического материала, проявляющего истинную нелинейную чувствительность к скорости изменения размера и формы. Представляется численный пример чувствительности к скорости деформации кольца из мягкой стали, который сравнивается с решением подобного же вопроса в котором игнорировалась чувствительность к скорости изменения формы и размера. Различия подчеркивают, что эффект чувствительности к скорости деформации может быть очень значительным. Выводы этого исследования подтверждают метод приближенных вычислений, выдвинутый Перроном.